Set-Valued Retractions and Subdivision Algorithms for Approximating Feasible Sets

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Abstract

A classical retraction appearing in topology is a continuous function $r : X \to Y$ mapping X to a subset Y of X such that the idempotency condition $r(y) = y$ holds for all elements $y \in Y$. Since Y can be implicitly defined via complicated constraints or properties in applications, the computation of its range Y is of general interest.

In [3] we study upper semi-continuous set-valued retractions $R : X \rightrightarrows Y$ with nonempty closed values and adapt a subdivision algorithm for reachable sets of differential inclusions in [2] to the computation of ranges. The metric projection computed by a nonlinear optimizer and the selection step of a subdivision method approximating the reachable sets are generalized to ranges of set-valued retractions.

For set-valued retractions we focus on the (weaker) invariance condition $y \in R(y)$ for all elements y in Y and not on the stronger condition $R(y) = \{y\}$. This condition fits better to a best approximation y of a given point x in Y and to fixed points $\hat{x} \in F(\hat{x})$ for a set-valued map $F : X \rightrightarrows Y$. The subdivision algorithm which was originally introduced in [1] starts from a bounding box G, uses a refinement step of the box collection in level k and decides in a selection step which refined boxes approximate the invariant or attractor set. In the selection process for ranges we check the intersection of the image of a refined box B for R with B via computing best approximations of test points in G. If nonempty, this classifies the case that the box has a nonempty intersection with the range of R.

Example classes of this family are the set-valued projection of compact sets, feasible sets of optimization problems, window retractions, special marginal maps, and the projection to the set of fixed points. A convergence result for the subdivision method shows that the union of all boxes from the box collection for increasing refinement level yields the range of a set-valued retraction.

Special examples show that set-valued retractions are more general than set-valued projections and can exist, if there are no existing continuous (single-valued) retractions so that the approximation of their ranges (if given implicitly via constraints or properties) are a challenging field of set-valued analysis.

References:

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